

# The BRST treatment of stretched membranes

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## Abstract

The BRST-invariant formulation of the bosonic stretched membrane is considered. In this formulation the stretched membrane is given as a perturbation around zero-tension membranes, where the BRST-charge decomposes as a sum of a string-like BRST-charge and a perturbation. It is proven, by means of cohomology techniques, that there exists to any order in perturbation theory a canonical transformation that reduces the full BRST-charge to the string-like one. It is also shown that one may extend the results to the quantum level yielding a nilpotent charge in 27 dimensions.

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# 1 Introduction

Membranes are interesting from many points of view, it may have a connection to M-theory [1, 2] and it is a generalization of the string action. In the lightcone gauge it can be reduced to a matrix model [3, 4, 5] which is conjectured to be M-theory [6]. It is probably also relevant as a D2-brane, being part of the strongly coupling region of string theory. The relevance of D-branes for string theory at strong coupling was first realized in [7]. It is also interesting by itself and as a testing ground to see if methods in string theory generalize to higher extended objects. Solutions of the equations of motions are rare because of the highly non-linear equations of motion.

In [8] we proposed to study so-called stretched membrane configurations. These are configurations which arise, in a partial fixing of the gauge, for weak tensions of the membrane. Stretched membranes may be treated perturbatively around a zero tension limit, which corresponds to a string-like theory. In [9] we proved that, by fixing the gauge completely to the lightcone gauge, there is a canonical equivalence between the two theories i.e. the membrane is, to any order in perturbation theory, equivalent to a string-like theory. Properties of stretched membranes may, therefore, be inferred from those of the string-like theory. The equivalence holds for bosonic as well as supersymmetric membranes. It was also shown that the canonical equivalence extends to a unitary one at the quantum level, yielding, among other results, the critical dimensions 27 and 11 for the bosonic and supersymmetric cases, respectively.

In this article we continue our analysis of the bosonic stretched membrane. The aim is to see whether the equivalence may be proven without the use of the lightcone gauge. Classically, one may argue that this must be the case, at least locally in phase-space. But at the quantum level this need not be true. Proving unitary equivalence for a fully gauged fixed theory does not, in general, imply that the same is true without gauge fixing, since the gauge symmetry may break down due to

anomalies. It is rather the converse that is true. By proving the unitary equivalence between the BRST-charge of the stretched membrane and the unperturbed string-like BRST-charge, we can conclude that, since the latter theory is non-anomalous, this is also true for the former one. From this it follows that we can impose any particular gauge and still maintain equivalence.

The problem to solve is, therefore, the following. Given a BRST-charge of the form

$$Q = Q^0 + Q', \tag{1.1}$$

where  $Q^0$  is the unperturbed string-like BRST-charge and  $Q'$  the perturbation, is it possible to find a canonical transformation which takes  $Q$  into  $Q^0$ ? Unfortunately, the techniques used in [9] do not readily generalize to the present case due to the complexity of the problem. Instead, we will use another approach. As we will see, it is possible to restate the problem as one of the cohomology of  $Q^0$ . If this cohomology is trivial for ghost number one then there will always exist, to any order in perturbation theory, a canonical transformation of the kind we are looking for. In fact, as we will show, the restatement of the perturbation problem as one in cohomology is not something particular for stretched membranes, but is quite general.

Since  $Q^0$  is essentially the BRST-charge of the bosonic string, the cohomology problem seems already to have been solved. This is not entirely correct. First of all, the basic fields including the ghosts, are fields defined on the world-volume rather than the world-sheet. Secondly, the known proofs of the cohomology of string theory do not directly apply to our case. The cohomology w.r.t. the quantum string state-space is well known [10, 11, 12, 13, 14]. Using the one-to-one correspondence between operators and states, one may also deduce the cohomology of the operators. In our case, we need to analyze the classical cohomology of the phase-space functions, which turns out to be a little bit different.

Having established the canonical equivalence one can turn to the quantum case. We will show that the quantization procedure proposed in [9] can, in a straightforward way, be applied to the present case. By this procedure one defines a specific ordering whereby the canonical transformations turn into unitary ones so that the equivalence of the perturbed and unperturbed theories is maintained at the quantum level. This then will show that quantum consistency, through the nilpotency of the membrane BRST-charge, requires 27 dimensions.

The paper is organized as follows. In section two we consider the BRST treatment of gauge theories formulated as perturbation theories. Here we also show the connection between the existence of canonical transformations and the BRST cohomology. In the third section we discuss the cohomology of the BRST-charge locally in phase-space. The cohomology problem relevant to the membrane is treated in section four. This will also show the canonical equivalence between the string-like theory and the stretched membrane. In the last section we discuss the quantization of our model.

## 2 BRST treatment of perturbatively formulated gauge theories

In this section we will, in more general terms, formulate the problem of finding canonical transformations which canonically map constraint theories formulated as an unperturbed and a perturbed part. We start with a general theory of this form. In our particular case, the unperturbed theory is a string-like theory, which is the standard bosonic string theory with extra world volume parameter dependence. The perturbed theory is the stretched membrane theory formulated in [8].

Consider the general situation where we have a theory with first-class constraints,

$\phi_a \approx 0$ , formulated as a perturbation theory

$$\phi_a[p_i, q^i] = \psi_a[p_i, q^i] + \sum_{n=1}^N g^n \lambda_a^{(n)}[p_i, q^i] \quad (2.1)$$

where  $g \ll 1$  is the perturbation parameter. Since we have a closed Poisson bracket algebra

$$\{\phi_a, \phi_b\} = U_{ab}{}^c \phi_c \quad (2.2)$$

to any order in  $g$ , it follows that the unperturbed part,  $\psi_a$ , also satisfies a closed algebra

$$\{\psi_a, \psi_b\} = U'_{ab}{}^c \psi_c, \quad (2.3)$$

where, in general,  $U_{ab}{}^c$  and  $U'_{ab}{}^c$  can depend on the phase-space variables and

$$U'_{ab}{}^c \equiv U_{ab}{}^c |_{g=0}. \quad (2.4)$$

The BRST-charge is generally of the form

$$Q = \sum_{n=0}^{(n)} Q, \quad (2.5)$$

where

$$\begin{aligned} Q^{(0)} &= \phi_a c^a \\ Q^{(n)} &= A_{a_1, \dots, a_{n+1}}^{b_1, \dots, b_n} c^{a_1} \dots c^{a_{n+1}} b_{b_1} \dots b_{b_n}, \end{aligned} \quad (2.6)$$

and the functions,  $A_{a_1, \dots, a_{n+1}}^{b_1, \dots, b_n}$ , are determined by the Poisson bracket algebra of the constraints and the nilpotency condition on the BRST-charge.

In the assumed perturbation theory we can also expand in terms of  $g$

$$Q = Q^0 + \sum_{i=1}^{N'} g^i Q^i. \quad (2.7)$$

The nilpotency condition of the full BRST-charge now yields relations to each order in  $g$ . The zeroth order relation gives that  $Q^0$ , the BRST-charge for the unperturbed theory, is nilpotent. To first order in  $g$  we have

$$\{Q^0, Q^1\} = 0. \quad (2.8)$$

Thus, we know that  $Q^1$  is in the cohomology of  $Q^0$ . If  $Q^1$  is a trivial element in this cohomology then there exists a function  $G_1$  satisfying

$$Q^1 = -\{Q^0, G_1\}. \quad (2.9)$$

Let us assume that  $G_1$  exists. Then we are free to interpret  $G_1$  as a generator of an infinitesimal canonical transformation. This transformation shifts  $Q$  to

$$Q \xrightarrow{G_1} Q^0 + g^2 \left( Q^2 - \frac{1}{2} \{ \{Q^0, G_1\}, G_1 \} \right) + \dots \quad (2.10)$$

The nilpotency condition to second order in  $g$  is

$$\left\{ Q^0, Q^2 - \frac{1}{2} \{ \{Q^0, G_1\}, G_1 \} \right\} = 0. \quad (2.11)$$

This implies that  $Q'^2 \equiv Q^2 - \frac{1}{2} \{ \{Q^0, G_1\}, G_1 \}$  is in the cohomology of  $Q^0$  and we have a problem of the same type as in eq. (2.8). If  $Q'^2$  is a trivial element in the cohomology, then we may repeat the above argument and conclude that there exists an infinitesimal canonical transformation that transforms  $Q$  to  $Q^0$  to second order in  $g$ . One may continue this to any order in  $g$ . Thus, we see that the problem of proving that there exists a canonical transformation to any order in perturbation theory may be solved by proving that the  $Q^0$  cohomology at ghost number one is trivial.

It should be remarked that the above argument goes through for the quantum case as well. Replacing all Poisson brackets with commutators shows that the problem of unitary equivalence can be restated in terms of the cohomology of the BRST-operator. For the stretched membrane, however, this is not helpful. The

argument requires that one can establish the nilpotency at the quantum level of the BRST-operator and this we cannot do from the outset. Instead we will have to proceed through the classical analysis and, using this, define a quantum theory by promoting the canonical transformations to unitary ones. The nilpotency will then follow as a consequence of the unitary equivalence.

### 3 Local existence of a canonical transformation

In this section we will continue to consider the general situation, but only locally in phase-space. We will show that in this case there always exists a canonical transformation of the type discussed above.

The starting point is the again a theory as in section two with constraints  $\phi_a$ . We will here first use the abelization theorem [15] (for a short proof of it, see [16]). The theorem states that for all constraint theories there exists, locally in phase-space, an invertible coordinate dependent matrix  $K_a{}^b$  such that  $F_a \equiv K_a{}^b \phi_b$  are abelian. For explicit constructions for the free bosonic string theory, see [17, 18]. A theorem by Henneaux [19] shows that there exists a canonical transformation,  $G$ , in the extended phase-space such that the BRST-charge of the unperturbed theory is canonically equivalent to an abelian one,

$$Q^0 \xrightarrow{G} \tilde{Q}^0 = F_a c^a. \quad (3.1)$$

Applying this canonical transformation to the full theory yields a BRST-charge

$$Q \xrightarrow{G} \tilde{Q} = \tilde{Q}^0 + \sum_{i=1}^{N'} g^i \tilde{Q}^i \quad (3.2)$$

where  $\tilde{Q}^0$  is given by eq. (3.1).

As we discussed in the previous section, the existence of a canonical transformation, which maps the perturbed BRST-charge to the unperturbed one, is determined

by the cohomology of the unperturbed BRST-charge. Let us, therefore, study the cohomology of the simple abelian model in more detail.

Assume that there exist  $m$  first class constraints,  $F_a \approx 0$ , in a theory with  $n$  degrees of freedom. Since the theory is abelian there exists locally, by Darboux's theorem, a canonical transformation from  $(q^i, p_i)$  to  $(\chi^a, q^{*j}, F_a, p_j^*)$ , where  $j = 1, \dots, n - m$  and  $\{\chi^a, F_b\} = \delta_b^a$ . The BRST-charge for the abelian model is

$$Q_A = F_a c^a. \quad (3.3)$$

One can also add ghost momenta,  $b_a$ , that satisfy

$$\{c^a, b_b\} = \delta_b^a. \quad (3.4)$$

We will restrict our study of the BRST cohomology to the space of polynomials in the phase-space coordinates. The proof we will give will not depend on the assumption of locality. This will be important as we will need the result in the next section, where the treatment is not restricted to being local.

Let us construct  $m$  charges from  $Q_A$  (no summation over  $a$ )

$$\begin{aligned} N_a &\equiv \{Q_A, \chi^a b_a\} \\ &= \chi^a F_a - c^a b_a. \end{aligned} \quad (3.5)$$

A non-trivial BRST-invariant function has to have zero eigenvalues, in the Poisson bracket sense, w.r.t. any of these charges. Otherwise, if a BRST-invariant function  $\mathcal{O}$  satisfies

$$\{N_a, \mathcal{O}\} = n_a \mathcal{O}, \quad (3.6)$$

it is BRST-trivial

$$\mathcal{O} = \frac{1}{n_a} \{Q_A, \{\mathcal{O}, \chi^a b_a\}\}. \quad (3.7)$$

The fundamental fields in this theory with non-zero eigenvalues of  $N_a$  are  $(F_a, b_a)$  with eigenvalue  $+1$  and  $(\chi^a, c^a)$  with eigenvalue  $-1$ . Thus, non-trivial BRST-invariant polynomials can depend on  $(\chi^a, F_a, c^a, b_a)$  only through the combinations



$(F_a c^a, F_a \chi^a, b_a c^a, b_a \chi^a)$  (no summation over  $a$ ). Define these linear combinations (no summation over  $a$ )

$$\begin{aligned} s_a &\equiv \frac{1}{2} b_a \chi^a \\ t_a &\equiv \frac{1}{2} (b_a c^a + F_a \chi^a) \\ u_a &\equiv \frac{1}{2} (b_a c^a - F_a \chi^a) \\ v_a &\equiv F_a c^a. \end{aligned} \tag{3.8}$$

They satisfy (no summation over  $a$ )

$$s_a \xrightarrow{Q_A} t_a \xrightarrow{Q_A} 0 \tag{3.9}$$

$$u_a \xrightarrow{Q_A} v_a \xrightarrow{Q_A} 0 \tag{3.10}$$

$$s_a^2 = v_a^2 = 0 \tag{3.11}$$

$$t_a^2 - u_a^2 = 2s_a v_a \tag{3.12}$$

$$(u_a + t_a)v_a = 0. \tag{3.13}$$

Eqs. (3.11) and (3.12) imply that we can reduce any polynomial to be at most linear in  $s_a$ ,  $v_a$  and  $u_a$ .

Let us determine the cohomology of the BRST-charge by first fixing to a generic value of  $a$  and suppress the indices of the fields  $(s_a, v_a, u_a, t_a)$ . Let  $f(s, t, u, v, x)$  be a BRST-invariant function where  $x$  indicates dependence on other fields. Expand first the  $s$ -dependence of  $f$

$$f = s f_1(t, u, v, x) + f_2(t, u, v, x). \tag{3.14}$$

The BRST-invariance of  $f$  implies

$$\{Q_A, f_1\} = 0 \tag{3.15}$$

$$\{Q_A, f_2(t, u, v, x)\} + t f_1(t, u, v, x) = 0. \tag{3.16}$$

The second equation implies that  $f_2$  can be split into two parts

$$f_2 = t G_1(t, u, v, x) + f_3(t, u, v, x) \tag{3.17}$$

where

$$\begin{aligned}\{Q_A, G_1(t, u, v, x)\} &= -f_1 \\ \{Q_A, f_3(t, u, v, x)\} &= 0.\end{aligned}$$

This is always possible, because otherwise  $f_1 = 0$ . Inserting eq. (3.17) into eq. (3.14) yields

$$\begin{aligned}f &= -s\{Q_A, G_1\} + tG_1(t, u, v, x) + f_3(t, u, v, x) \\ &= \{Q_A, sG_1\} + f_3(t, u, v, x).\end{aligned}\tag{3.18}$$

Thus, non-trivial functions in the cohomology of  $Q_A$  are independent of  $s$ . We can expand  $f_3$  as

$$f_3 = uvf_4^1(t, x) + uf_4^2(t, x) + f_5(t, v, x).\tag{3.19}$$

The BRST-invariance of  $f_3$  implies

$$\begin{aligned}\{Q_A, f_4^1(t, x)\} &= 0 \\ \{Q_A, f_4^2(t, x)\} &= 0 \\ \{Q_A, f_5(t, v, x)\} + vf_4^2(t, x) &= 0.\end{aligned}\tag{3.20}$$

The first equation shows us that  $uvf_4^1$  is trivial

$$\begin{aligned}\{Q_A, vsf_4^1\} &= -tvf_4^1, \\ &= uvf_4^1,\end{aligned}\tag{3.21}$$

where the last equality follows from eq. (3.13). Eq. (3.20) also implies that one can split  $f_5$  into two parts

$$f_5 = vG_2(t, x) + f_6(t, v, x),\tag{3.22}$$

with

$$\begin{aligned}\{Q_A, G_2\} &= f_4^2 \\ \{Q_A, f_6\} &= 0.\end{aligned}\tag{3.23}$$

Extracting the  $v$ -dependence of the function  $f_6$ ,

$$f_6 = v f_7(t, x) + f_8(t, x), \quad (3.24)$$

shows us that the BRST-invariance of  $f$  implies that both functions,  $f_7$  and  $f_8$ , are BRST-invariant. The first term is BRST-trivial,  $v f_7 = \{Q_A, u f_7\}$ . Expanding the  $t$ -dependence of  $f_8$

$$f_8 = \sum_{j=0}^{\infty} t^j h_j(x), \quad (3.25)$$

yields that each  $h_j$  has to be BRST-invariant. This implies that

$$f_8 = \{Q_A, G_3\} + h_0(x), \quad (3.26)$$

where

$$G_3 = \sum_{j=1}^{\infty} s t^{j-1} h_j(x). \quad (3.27)$$

Collecting all parts we have

$$f = \{Q_A, s G_1 + u G_2 + v s f_4^1 + u f_7 + G_3\} + h_0(x). \quad (3.28)$$

Concluding, we have shown that all non-trivial phase-space polynomials in the cohomology are independent of  $t_a$ ,  $u_a$ ,  $v_a$  and  $s_a$ , for a fixed value of  $a$ . This is true for all values of  $a$ . Thus, the only non-trivial elements in the cohomology are ghost number zero polynomials that only depend on  $(q^{*j}, p^{*j})$ , the coordinates that span the physical phase-space.

Let us now return to our problem of proving the existence of a canonical transformation. From our results of the cohomology and from the previous section we have proven that such a transformation exists to all orders in perturbation theory. This implies that we have proven the existence of the canonical transformations  $G$  and  $G'$  such that

$$Q \xrightarrow{G} \tilde{Q} \xrightarrow{G'} \tilde{Q}^0 \xrightarrow{G^{-1}} Q^0, \quad (3.29)$$

where  $G$  transforms the unperturbed constraints to abelian ones,  $G'$  transforms the perturbed BRST-charge to the unperturbed abelian one and, finally,  $G^{-1}$  transforms us to the original unperturbed BRST-charge. These statements are true locally. There may still, however, exist obstructions preventing the results to hold globally.

## 4 Application to the stretched membrane

We have seen from the previous section that we are assured that there exists, at least locally, a canonical transformation transforming the full BRST-charge to the unperturbed one. We will now consider what happens in the specific case of the stretched membrane when we do not restrict ourselves to local considerations. Although the results of the local case will not be needed as such, we do need to use the result from the analysis of the abelian BRST-charge, which was not restricted to be local in phase-space.

For the stretched membrane the unperturbed BRST-charge is of the form eq. (1.1) where  $Q^0$  is that of a free string with an extra world-parameter dependence. The cohomology of the state-space of the free quantum string theory is well known, see [10, 11, 12, 13, 14]. The cohomology of the classical theory has, however, not to our knowledge been solved. Using techniques largely based on [14], we will analyse the cohomology of our string-like model.

If we reduce one of the three constraints for the membrane theory and introduce two ghosts and ghost momenta for the remaining constraints, one can construct a BRST-charge for this theory<sup>3</sup>

$$Q = \int d^2\xi \mathcal{Q} \tag{4.1}$$

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<sup>3</sup>We have corrected a sign error in [8]

$$\begin{aligned}
\mathcal{Q} &= \phi_1 c^1 + \phi_2 c^2 + \partial_1 c^1 c^1 b_1 + \partial_1 c^2 c^2 b_1 + \partial_1 c^2 c^1 b_2 + \partial_1 c^1 c^2 b_2 \\
&+ g \left[ \mathcal{P} \partial_2 X \partial_2 c^1 c^2 b_1 - \partial_1 X \partial_2 X \partial_2 c^2 c^2 b_1 + (\partial_2 X)^2 \partial_1 c^2 c^2 b_1 \right. \\
&+ \left. 2\mathcal{P} \partial_2 X \partial_2 c^2 c^2 b_2 - 2\partial_2 c^1 \partial_2 c^2 c^2 b_1 b_2 \right], \tag{4.2}
\end{aligned}$$

where

$$\begin{aligned}
\phi_1 &= \mathcal{P} \partial_1 X \\
\phi_2 &= \frac{1}{2} \left\{ \mathcal{P}^2 + (\partial_1 X)^2 \right. \\
&+ \left. g \left[ (\partial_1 X)^2 (\partial_2 X)^2 + (\mathcal{P} \partial_2 X)^2 - (\partial_1 X \partial_2 X)^2 \right] \right\}. \tag{4.3}
\end{aligned}$$

We can split this BRST-charge into two parts, one free part, which is that of a string-like theory, and a perturbation

$$Q = Q^0 + gQ^1. \tag{4.4}$$

If we make a change of variables from  $(X^\mu, \mathcal{P}_\mu)$ ,  $\mu = 0, \dots, D-2$ , to the Fourier coefficients  $(q_n^\mu, \tilde{q}_n^\mu, \alpha_{m,n}^\mu, \tilde{\alpha}_{m,n}^\mu)$ , we have the non-zero Poisson brackets

$$\begin{aligned}
\{\alpha_{m,n}^\mu, \alpha_{p,q}^\nu\} &= \{\tilde{\alpha}_{m,n}^\mu, \tilde{\alpha}_{p,q}^\nu\} = -im\eta^{\mu\nu} \delta_{m+p,0} \delta_{n+q,0} \\
\{q_m^\mu, \alpha_{0,n}^\nu\} &= \{\tilde{q}_m^\mu, \tilde{\alpha}_{0,n}^\nu\} = \eta^{\mu\nu} \delta_{m+n,0} \tag{4.5}
\end{aligned}$$

To simplify the equations, we redefine our ghosts and ghost momenta

$$\begin{aligned}
c &= c^1 + c^2 \\
\tilde{c} &= c^1 - c^2 \\
b &= \frac{1}{2} (b_1 + b_2) \\
\tilde{b} &= \frac{1}{2} (b_1 - b_2). \tag{4.6}
\end{aligned}$$

Fourier expanding these fields we find the non-zero Poisson brackets

$$\{c_{m,n}, b_{p,q}\} = \{\tilde{c}_{m,n}, \tilde{b}_{p,q}\} = \delta_{m+p,0} \delta_{n+q,0}. \tag{4.7}$$

Choose lightcone coordinates

$$\begin{aligned} A^+ &= \frac{1}{\sqrt{2}} (A^{D-2} + A^0) \\ A^- &= \frac{1}{\sqrt{2}} (A^{D-2} - A^0) \end{aligned} \quad (4.8)$$

and introduce a grading by

$$\begin{aligned} N_{lc} &= \sum_{m \neq 0, n} \frac{1}{im} (\alpha_{-m, -n}^+ \alpha_{m, n}^- + \tilde{\alpha}_{-m, -n}^+ \tilde{\alpha}_{m, n}^-) + \sum_{n \neq 0} (\alpha_{0, -n}^- q_n^+ + \tilde{\alpha}_{0, -n}^- \tilde{q}_n^+ \\ &\quad - \alpha_{0, -n}^+ q_n^- - \tilde{\alpha}_{0, -n}^+ \tilde{q}_n^-) + \alpha_{0, 0}^- q_0^+ + \tilde{\alpha}_{0, 0}^- \tilde{q}_0^+. \end{aligned} \quad (4.9)$$

$N_{lc}$  acts diagonally, within Poisson brackets, on the basic fields.  $q_{n \neq 0}^-$ ,  $\tilde{q}_{n \neq 0}^-$ ,  $\alpha_{m, n}^-$  and  $\tilde{\alpha}_{m, n}^-$  have eigenvalue  $+1$ ;  $q_n^+$ ,  $\tilde{q}_n^+$  for all  $n$ ,  $\alpha_{m, n}^+$  and  $\tilde{\alpha}_{m, n}^+$  for  $|m| + |n| \neq 0$  have eigenvalue  $-1$ . All other fields have eigenvalue zero.

The string-like BRST-charge may now be split into two parts

$$Q^0 = Q_1 + Q_0, \quad (4.10)$$

where the lower index indicates the eigenvalue w.r.t.  $N_{lc}$ . The nilpotency of  $Q^0$  implies

$$\{Q_1, Q_1\} = \{Q_0, Q_1\} = \{Q_0, Q_0\} = 0. \quad (4.11)$$

Thus, the two separate terms in  $Q^0$  are nilpotent by themselves. The explicit form of  $Q_1$  is simple

$$Q_1 = \sum_{m, n} (\alpha_{0, 0}^+ \alpha_{m, n}^- c_{-m, -n} + \tilde{\alpha}_{0, 0}^+ \tilde{\alpha}_{m, n}^- \tilde{c}_{-m, -n}), \quad (4.12)$$

and it is the BRST-charge of an abelian theory. One may, as we will see below, use  $Q_1$  to study the BRST-cohomology of the full theory. This requires us to determine the  $Q_1$ -cohomology, which we can do using the analysis of the abelian case given in the previous section. In order to apply this analysis we need the existence of gauge fixing functions  $\chi_{m, n}$  and  $\tilde{\chi}_{m, n}$  such that  $\{\chi_{m, n}, Q_1\} = c_{m, n}$  and  $\{\tilde{\chi}_{m, n}, Q_1\} = \tilde{c}_{m, n}$ .

Such functions exist if we assume that  $\alpha_{0,0}^+$  and  $\tilde{\alpha}_{0,0}^+$ , which are conserved quantities, are nonzero. Then  $\chi_{m,n} = \frac{i}{m\alpha_{0,0}^+}\alpha_{m,n}^+$  for  $m \neq 0$  and  $\chi_{0,n} = \frac{1}{\alpha_{0,0}^+}q_n^+$  etc. for  $\tilde{\chi}_{m,n}$ . We will, in analyzing the cohomology, only consider functions that are finite degree polynomials in the basic fields, except  $\alpha_{0,0}^+$  and  $\tilde{\alpha}_{0,0}^+$ , where we permit inverse powers as well. Furthermore, we will assume no dependence on  $q_0^-$  and  $\tilde{q}_0^-$ , which is sufficient for our case.

We can now proceed and use the results of the previous section. This yields that the non-trivial polynomials in the cohomology of  $Q_1$  have zero ghostnumber and have the dependence

$$h_{\{Q_1\}} = h_{\{Q_1\}} \left( q_n^I, \tilde{q}_n^I, q_{n \neq 0}^-, \tilde{q}_{n \neq 0}^-, \alpha_{m,n}^I, \tilde{\alpha}_{m,n}^I, \alpha_{0,n}^+, \tilde{\alpha}_{0,n}^+ \right). \quad (4.13)$$

where  $I = 1, \dots, D-3$ . Let us now study the cohomology of the string-like BRST-charge. One may expand a general BRST-invariant polynomial,  $K$ , in terms of its eigenvalues of  $N_{lc}$  defined in eq. (4.9)

$$K = K_N + K_{N-1} + \dots + K^I, \quad (4.14)$$

where

$$\{N_{lc}, K_n\} = nK_n. \quad (4.15)$$

By assumption,  $N$  and  $I$  are finite. The BRST-invariance implies

$$\begin{aligned} 0 = \{Q, K\} &= \overbrace{\{Q_1, K_N\}}^0 + \overbrace{\{Q_0, K_N\} + \{Q_1, K_{N-1}\}}^0 + \dots \\ &+ \overbrace{\{Q_0, K_{N-i+1}\} + \{Q_1, K_{N-i}\}}^0 + \dots + \overbrace{\{Q_0, K_I\}}^0, \end{aligned} \quad (4.16)$$

where  $i = 1, 2, \dots$ . Thus, the highest order term,  $K_N$ , is BRST-invariant w.r.t.  $Q_1$ . Using the cohomology of  $Q_1$ , there exists two possibilities. Either  $K$  has a ghost number different from zero which, by our analysis of the  $Q_1$ -cohomology, implies that the highest order term is BRST-trivial. This in turn implies, by the

same reasoning, that all other terms with lower eigenvalue of  $N_{lc}$  are trivial as well. Another possibility is that  $K$  has zero ghost number. Although this case is not needed for our problem, we consider it out of general interest. For zero ghost number there can exist a non-trivial part in  $K_N$

$$K_N = h_N^{(N)} + \{Q_1, C_{N-1}\}, \quad (4.17)$$

where  $h_N^{(N)}$  is a non-trivial function in the cohomology of  $Q_1$  and  $C_{N-1}$  has ghost number  $-1$  and eigenvalue  $(N-1)$  of  $N_{lc}$ . The phase-space function  $h_N^{(N)}$  depends, by the analysis of the cohomology of  $Q_1$ , only on the fields  $(q_n^I, \tilde{q}_n^I, q_{n \neq 0}^-, \tilde{q}_{n \neq 0}^-, \alpha_{m,n}^I, \tilde{\alpha}_{m,n}^I, \alpha_{0,n}^+, \tilde{\alpha}_{0,n}^+)$ . Inserting eq. (4.17) into the equation for the next order yields

$$\{Q_0, h_N^{(N)}\} + \{Q_1, K_{N-1} - \{Q_0, C_{N-1}\}\} = 0. \quad (4.18)$$

One can split  $K_{N-1}$  into two parts  $h_{N-1}^{(N)} + K'_{N-1}$  such that

$$\{Q_1, h_{N-1}^{(N)}\} = -\{Q_0, h_N^{(N)}\}. \quad (4.19)$$

This equation can always be solved since the right-hand side is  $Q_1$ -exact and has ghost number equal to one. Thus, from the  $Q_1$ -cohomology, there will always exist a function  $h_{N-1}^{(N)}$ . Eq. (4.18) now implies

$$\{Q_1, K'_{N-1} - \{Q_0, C_{N-1}\}\} = 0, \quad (4.20)$$

which is of a similar form as the equation previously solved. Consequently, the solution to  $K_{N-1}$  is

$$K_{N-1} = h_{N-1}^{(N)} + h_{N-1}^{(N-1)} + \{Q_1, C_{(N-2)}\} + \{Q_0, C_{(N-1)}\}, \quad (4.21)$$

where  $h_{N-1}^{(N-1)}$  is a function of  $(q_n^I, \tilde{q}_n^I, q_{n \neq 0}^-, \tilde{q}_{n \neq 0}^-, \alpha_{m,n}^I, \tilde{\alpha}_{m,n}^I, \alpha_{0,n}^+, \tilde{\alpha}_{0,n}^+)$ . One can proceed in the same way to any order in  $N_{lc}$ . This yields the same kind of equations and the result in the end is

$$K = \sum_{i=I}^N \sum_{j=i}^N h_j^{(i)} + \{Q^0, C\}, \quad (4.22)$$



where we have defined

$$C \equiv \sum_{i=I}^N C_{i-1}. \quad (4.23)$$

The functions  $h_i^{(j)}$ , where  $i \leq N$  and  $j \leq i$ , are determined from the term with the highest eigenvalue of  $N_{lc}$ , thus, by  $h_i^{(i)}$ . This term only depends on the fields  $(q_n^I, \tilde{q}_n^I, q_{n \neq 0}^-, \tilde{q}_{n \neq 0}^-, \alpha_{m,n}^I, \tilde{\alpha}_{m,n}^I, \alpha_{0,n}^+, \tilde{\alpha}_{0,n}^+)$ . Collecting the terms  $h_i^{(j)}$ , we can construct functions that are BRST-invariant and non-trivial w.r.t. the full string-like BRST-charge

$$h^{(j)} \equiv \sum_{i=-I}^j h_i^{(j)}. \quad (4.24)$$

These functions are such that the term which has the highest value w.r.t.  $N_{lc}$  is non-trivial in the cohomology of  $Q_1$  and terms with lower eigenvalue, are correction terms such that the function is in the cohomology of  $Q^0$ .

Let us now conclude the analysis of the cohomology of the string-like BRST-charge. We have found that in the space of finite degree polynomials of the basic fields, excluding dependence on  $q_0^-$  and  $\tilde{q}_0^-$ , and assuming  $\alpha_{0,0}^+$  and  $\tilde{\alpha}_{0,0}^+$  to be non-zero as they enter in the expressions with inverse powers, the cohomology is non-trivial only for zero ghost number. We may now use the result of section two to conclude that, provided our assumptions are valid, the membrane BRST-charge is canonically equivalent to the string-like one.

Considering our assumptions, we have first of all the restriction to finite degree polynomials. This is always true within our perturbation theory. Secondly, the assumption that  $\alpha_{0,0}^+$  and  $\tilde{\alpha}_{0,0}^+$  are non-zero is basically the same assumption one has for the known proof of the cohomology of string theory. As these fields are conserved quantities, this restricts possible initial conditions.

The final assumption, namely the exclusion of  $q_0^-$  and  $\tilde{q}_0^-$  dependence, requires some more elaborate discussion. The zeroth and first order perturbation does not involve  $q_0^-$  and  $\tilde{q}_0^-$ . This implies, by the proof of the cohomology in the abelian case

in the previous section, that there exists an infinitesimal canonical transformation to first order. Using the gradation w.r.t.  $N_{lc}$  defined in eq. (4.9), one may construct the generator of the canonical transformation order by order in  $N_{lc}$ . It is straightforward to see that this generator will not depend on  $q_0^-$  and  $\tilde{q}_0^-$ , which in turn implies that no higher order terms that are generated will depend on these fields, either. We can proceed in this way order by order proving the assertion.

## 5 Quantization

We will in this section discuss the quantization of our model. This is done in the same manner as in the lightcone formulation in [9]. We will, therefore, only repeat the essential features and discuss the differences of the two formulations.

We have in the previous section proven that there exists, to any order in perturbation theory, a canonical transformation connecting the stretched membrane model to the free string-like theory. We will now define the quantum theory for the stretched membrane from the free string-like theory by lifting the canonical transformations to unitary ones.

We define the unitary transformations by an iterative procedure. At some arbitrary order  $N$  in perturbation theory we define a unitary operator

$$U_N \equiv \exp(-i :_{N-1} G_N :_{N-1}). \quad (5.1)$$

Here  $G_N$  is the  $N$ 'th order contribution to the generator of infinitesimal transformations, which we, from the previous section, know exists classically. At the quantum level we specify the corresponding operator by the ordering  $:_{N-1}$ , which is the normal ordering w.r.t. the  $(N-1)$ 'th order vacuum. This vacuum is defined by

$$|0, k^+ \neq 0\rangle_{N-1} = U_{N-1} \cdot \dots \cdot U_1 |0, k^+ \neq 0\rangle_0, \quad (5.2)$$

where the zeroth order vacuum,  $|0, k^+ \neq 0\rangle_0$ , is defined in the usual way. Note that the condition  $k^+ \neq 0$  is slightly different from the one in the lightcone formulation

in [9]. The full unitary transformation to order  $N$  in perturbation theory is then

$$U^{(N)} = U_N \cdot \dots \cdot U_1. \quad (5.3)$$

From the vacuum it is straightforward to construct the physical states for the stretched membrane theory to any finite order in perturbation theory. This is done in the same way as in the lightcone formulation. As an example, the new oscillators to order  $N$  are defined as

$$\alpha_{m,n}^{(N),\mu} \equiv U_N \cdot \dots \cdot U_1 \alpha_{m,n}^\mu U_1^\dagger \cdot \dots \cdot U_N^\dagger. \quad (5.4)$$

Through our construction it follows immediately that to any order  $N$  in perturbation theory

$$(Q)^2 = \frac{1}{2}[Q, Q] = \frac{1}{2}[U^{(N)}Q^0U^{(N)\dagger}, U^{(N)}Q^0U^{(N)\dagger}] = \frac{1}{2}[Q^0, Q^0] = 0, \quad (5.5)$$

where the last equality is true only for  $D = 27$ .

The partial gauge has singled out the  $(D - 1)$ -direction and the corresponding field components are given by

$$\begin{aligned} X^{D-1} &= \frac{1}{\sqrt{g}}\xi^2 \\ \mathcal{P}_{D-1} &= -\sqrt{g}B, \end{aligned} \quad (5.6)$$

where we have defined

$$B = \mathcal{P}_\mu \partial_2 X^\mu. \quad (5.7)$$

One of the relevant physical operators found in [9] involved the integrated lightcone version of  $B$ . If we integrate  $B$ , denote it by  $B_0$ , then it is gauge invariant, but not BRST-invariant. In order to construct a BRST-invariant expression one has to add ghosts to  $B_0$ . One will find the following BRST-invariant expression of  $B_0$ <sup>4</sup>

$$B_0 = \int d^2\xi \left\{ \mathcal{P}_\mu \partial_2 X^\mu - i\partial_2 cb - i\partial_2 \tilde{c}\tilde{b} \right\}. \quad (5.8)$$

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<sup>4</sup>To get this expression we have redefined the ghost momenta by a factor  $-i$ , such that one has the conventional anti-commutation relations

$B_0$  has the property that it is invariant under the constructed unitary transformations. This follows directly from the fact that  $B_0$  is an eigenvalue operator which counts the mode number in the  $\xi^2$ -direction, and that the net the mode number of the unitary operators are zero.

A final comment is that the BRST operator is only covariant w. r. t. the  $D - 1$ -dimensional subgroup of the full Lorentz group. Consequently, it is still an open question whether the full Lorentz group is anomaly free.

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